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NOTE

On the Domain of Divergence of Hermite–Fejér Interpolating Polynomials

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Recently, the study of the behavior of the Hermite–Fejér interpolants in the complex plane was initiated by L. Brutman and I. Gopengauz (1999, *Constr. Approx.* **15**, 611–617). It was shown that, for a broad class of interpolatory matrices on $[-1, 1]$, the sequence of polynomials induced by Hermite–Fejér interpolation to $f(z) \equiv z$ diverges everywhere in the complex plane outside the interval of interpolation $[-1, 1]$. In this note we amplify this result and prove that the divergence phenomenon takes place without any restriction on the interpolatory matrices.
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1. INTRODUCTION

Let $X = \{x_k^{(n)}\}$, $1 \leqslant k \leqslant n$, $n = 1, 2, ...,$ be an infinite triangular matrix, where

$$-1 \leqslant x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leqslant 1,$$

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and let $C[-1, 1]$ denote the Banach space of continuous functions on $[-1, 1]$ equipped with the uniform norm

$$\|f\| = \max_{x \in [-1, 1]} |f(x)|.$$

For $f \in C[-1, 1]$ there corresponds a unique interpolating polynomial $L_{n-1}(f; X; x)$ of degree at most $n-1$ coinciding with f at the nodes of the nth row of X, which can be expressed as

$$L_{n-1}(f; X; x) = \sum_{k=1}^{n} f(x_k^{(n)})\, l_k^{(n)}(X; x),$$

where, sometimes omitting the superfluous notations,

$$l_k(x) = l_k^{(n)}(X; x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)}, \qquad \omega_n(x) = \prod_{k=1}^{n} (x - x_k).$$

Furthermore, with every continuous function one can associate the so-called Hermite–Fejér interpolating polynomial $H_{2n-1}(f; X; x)$ of degree $2n-1$, defined by the following conditions (see, e.g., [4, Chap. 5]):

$$\begin{cases} H_{2n-1}(f; X; x_k) = f(x_k), & (k = 1, 2, ..., n), \\ H'_{2n-1}(f; X; x_k) = 0, & (k = 1, 2, ..., n). \end{cases}$$

One of the most important problems in interpolation theory is to characterize under what conditions on f and X the sequences $\{L_n(f; X; x)\}$ and $\{H_{2n-1}(f; X; x)\}$ converge to $f(x)$.

In the following we assume that the approximated function is not only continuous on $[-1, 1]$, but analytic in a domain of the complex plane containing $[-1, 1]$. It is well known (see, e.g., [3]) that for Lagrange interpolation the convergence behavior of the sequence $L_n(f; X; z)$ depends on the location of singular points. In particular, for entire functions, the convergence in the whole complex plane is guaranteed for any interpolatory matrix of nodes.

Very recently two of us considered an analogous problem for Hermite–Fejér interpolation [1] and showed that the behavior of the Hermite–Fejér interpolants in the complex plane is totally different. Namely, we proved that for the specific test function $f(z) \equiv z$, and for the important case of interpolatory matrices consisting of the roots of orthogonal polynomials, the sequence $H_{2n-1}(f; X; z)$ diverges everywhere outside of the interval of interpolation $[-1, 1]$.

In the present note we amplify this result, proving that the divergence phenomenon takes place without any restriction on the interpolatory matrices X.

2. RESULT

The goal of this note is to prove the following divergence result:

THEOREM 1. *Let X be an arbitrary interpolatory matrix of nodes in $[-1, 1]$ and let $f(z)=z$. Denote by $H_{2n-1}(f; X; z)$, $n=1, 2, ...,$ the corresponding sequence of Hermite–Fejér interpolation polynomials. Then, the approximation error $r_{2n-1}(z) := f(z) - H_{2n-1}(f; X; z)$, satisfies*

$$\limsup_{n\to\infty} |r_{2n-1}(z)| > 0, \qquad \forall z \notin [-1, 1]. \tag{1}$$

Proof. We need the following estimate which was proved in [1, p. 612]

$$|r_{2n-1}(z)| \geqslant \frac{\rho(z)}{|z^2-1|} \sum_{k=1}^{n} \frac{|\omega_n^2(z)|}{[\omega_n'(x_k)]^2}, \qquad z \notin [-1, 1], \tag{2}$$

where $\rho(z) := \operatorname{dist}(z, [-1, 1])$. As a direct consequence of (2) we have

$$|r_{2n-1}(z)| \geqslant \frac{\rho^3(z)}{|z^2-1|} \sum_{k=1}^{n} |l_k^2(z)|, \qquad z \notin [-1, 1]. \tag{3}$$

Thus, in order to show the divergence, it will be sufficient to prove that there exists $\delta > 0$ such that for every $z \notin [-1, 1]$ and sufficiently large n

$$S_n(z) := \sum_{k=1}^{n} |l_k^2(z)| > \delta. \tag{4}$$

Actually, we prove that $S_n(z) \geqslant 1/4$ for $n > 6$.

Let $z = x + iy$. In view of the definition of the fundamental polynomials and the triangle inequality

$$|z - x_k| \geqslant |x - x_k|, \qquad 1 \leqslant k \leqslant n, \qquad z \in \mathbb{C},$$

we have $|l_k(z)| \geqslant |l_k(x)|$. Hence,

$$S_n(z) = \sum_{k=1}^{n} |l_k^2(z)| \geqslant \sum_{k=1}^{n} l_k^2(x) := S_n(x). \tag{5}$$

Consider first $x \in [-1, 1]$. By Lemma IV in [2, p. 529], for $n > 6$ and a certain j (depending on x), $l_j(x) \geqslant 1/2$, so that

$$S_n(x) \geqslant l_j^2(x) \geqslant 1/4, \qquad n > 6. \tag{6}$$

Now let $x > 1$. Then, taking into account that $l_n(x_n) = 1$ and that $l_n(x)$ is monotone increasing for $x \geqslant x_n$, we have $l_n(x) > 1$, and thus

$$S_n(x) > 1. \tag{7}$$

Analogously, (7) holds for $x < -1$, thus completing the proof.

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